Visualisation in teaching-learning mathematics: the role of the computer

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Abstract

This paper is centred on the study of the role that information visualisation can have in mathematics learning. A theoretical analysis is worked out in order to give account of the complex phenomena that take place in the classroom in order to allow students to construct a mental image of the mathematical object involved in the activity. This theoretical analysis is the reference to study the different didactical functions that information visualisation can develop in learning process. This different functions are discussed making reference to the role that Cabri Gèométre can have in the teaching/learning processes of Euclidean geometry.

Keywords: *Mathematics learning, Information visualisation, Visual imagery, mathematical discourse* .

1. Introduction

We use the term visualisation to refer to the complex phenomena of visual imagery that plays a central role in all meaning and understanding as well as in all reasoning. About the nature of images that the mind forms following an external stimulus, many studies, also on contrasting lines, were developed in the last 20 years. Here we are interested in the phenomena of visual imagery that take place in maths learning and in particular in the dialectic that develops between dynamic external visual representations mediated by the technology (information visualisation) and visual imagery.

Speaking of external representations we mainly refer to two or three-dimensional representations of some aspects of a mathematical structure. Such representations may be static or dynamic as in the case of representations mediated by the computer. Dreyfus in [2] pointed out that the dialectic between external visual representation and visual imagery implies two mappings: from the mathematical structure to the visual representation and from the visual representation to the mental image. While the first mapping can be subject to mathematical analysis (epistemological analysis of the knowledge embedded in the structure of the visual representation), the second one is much more difficult to analyse, since there is no direct access to mental images. So this latter mapping can only be postulated on the basis of interpretations of how the external representations (graphics, signs, tables, drawings,…) are produced and used in the communicative context of the teaching/learning process and on the basis of the justifications that everyone gives of this way of use.

The two mappings evidenced by Dreyfus are important while studying the conditions under which computer-supported visualisation is useful for mathematics learning.

This is true both for the design of new systems for mathematics education and for the analysis of how systems already available on the market can be used in the classroom.

In this work these two mappings will be discussed, analysing the roles assumed by information visualisation [7] in the teaching/learning processes of mathematics. This analysis will be carried out on the basis of the experience we have developed in the design, implementation and experimental evaluation of computer-based visualisation systems for mathematics learning [1].

2. Mathematical objects, visual representation, mental image

For the aims of our work we consider Mathematics as a domain of knowledge that is concerned with “mathematical objects”, that is to say with objects with certain specific properties. Mathematical objects are abstract objects; indeed mathematics objects are not amenable to any concrete imagination or manipulation; they are immaterial, not tangible and accessible only to our thinking

In mathematics learning, differently from the physical concrete world, the learning object cannot be shown in an obstensive way, can be only conjured up by means of the use of external representations. There is not the possibility of directly accessing that “thing” that we can suppose to be the meaning of the representations.

Mathematical concepts such as numbers, functions, vectors, (which are not objects in a usual manner, but which embody relationships) are not directly accessible through everyday experience nor within intuitive perception, as for instance real or physical objects are, but they have to be represented by signs or symbols.

This is true also in the case of Euclidean geometrical learning (as we will see in section 4), where the perception involved in managing external representations (drawings) can be also an obstacle for the construction of a mental image, theoretically founded, of the correspondent geometrical object (figure).
Representations and symbols of mathematics establish a semiotic system which is of fundamental importance for any mathematical activity.
According to this epistemological position mathematical knowledge is not simply a ready made product that can be directly introduced into processes of teaching and learning. The new mathematical knowledge will only be actively constructed, in social interaction, by the student in his or her learning process within an activity.
For example, in the approach to rational numbers the relationship that takes place between representation and the mathematical object which it refers to (rational numbers), is very complex. The discourse about rational numbers suggests that these abstract objects are unique entities; on the other hand we have various representations for the rational numbers: the common fraction symbol, points on the number line, materialisation of different kinds, classes of pairs of integers; these material objects of representations do not enjoy all the properties we attribute to rational numbers [3].
It is not easy for students to understand that 2/3 and 4/6 are two different fractions which are quivalent since they refer to the same object, that is to say to the same rational number. Showing that 2/3 is obtained from 4/6 dividing the numerator and the denominator by 2 gives us a method that permits us to justify the conditions under which the two fractions are equivalent but we cannot work out an obstensive test of such an object and of its properties. We do not have imaginative access to anything which we could consider an image of the rational number represented by 2/3 or 4/6.
The image of this mathematical object emerges in the dialectic between understanding and expression relating to the mathematical activity in which the participants are involved. This image entails an orientation to negotiations with oneself about meaning, something that is outside the experience of school students [6].
The image of the mathematical object is strictly linked to the image of the mathematical activity that is negotiated by the participants in school practice. It is conditioned by social interaction; on one hand the structure of the external representations provide meaning for the mathematical discourse about the mathematical object involved in the activity and on the other hand the mathematical discourse contributes to structure an image of the mathematical activity, that is to say it contributes to structure an image of the object of this activity, that is a mathematical object. It is important to observe that mathematical discourse that emerges in social interaction within the activity is qualitatively different from any representation.
Understanding what a mathematical object is depends crucially on a very specific way of viewing and treating representations and their related mental images within a mathematical discourse.
In other words, taking into account school practice in which students and teacher are involved, the relationships between external visual representation, mathematical discourse and mental images can be sketched in this way.
Within an activity, the structure of external representations mediate the possibility to develop a meaningful mathematical discourse about the properties of the mathematical object starting from how the properties of the external representation are used in practice.
Mathematical discourse (which is always mediated by the reference to the mathematical object) allows a metaphoric use of the concrete meaning developed by working with external representations within the structure of an activity. It allows us to associate interpretations to the external representation used in the activity which can be justified on the basis of actions and goals as actually generated within an activity (in relation to some problem or task) and on the basis of cultural and historical considerations on the value and properties of the mathematical object involved in the activity. In this way the mathematical discourse contributes to transform an external representation into a mental image of a mathematical object, that is to say into an image of the abstract object which transcends the structure and the characteristics of the external representation.
From a psychological point of view the dialectic between external representation and mathematical discourse mediates the possibility for the subject to control, on the basis of external stimuli (external representations), the accordance of the mental image of the mathematical object involved in the activity with the shared ideas of the society.
Hence according to [3] grasping the properties of a mathematical object is always the result of interplay between visual and diagrammatic aspects involved in managing external representations and prepositional aspects of the discourse with respect to the mathematical object involved in the activity.

3. Information visualisation and mathematics learning

The design or the use of a computer-based system for mathematics learning requires careful consideration of the conditions under which the characteristics of form and interactivity of a system can develop, within an activity, a dynamic relationship between external representation and mental image which is effective for learning.

According to [5] we use the term information visualisation to intend the use of computer supported, interactive, visual representation of abstract data to amplify cognition. This definition is particularly appropriate when it refers to the teaching and learning processes of mathematics. As previously pointed out, mathematical objects are not amenable to any visual perception or manipulation. The traditional approach to mathematics knowledge is a symbolic re-constructive approach and it is developed inside the interaction between the student and the teacher, usually according to a transmissive teaching strategy. In this approach students have few opportunities of exploring the functionality of the symbolic representation at hand and of reflecting on the properties and characteristics of its structure. The prescriptive character of the mathematical discourse that emerges in the classroom focuses the attention on the system of rules attached to the symbolic representation at hand and leaves in the background the construction of a mental image of the mathematical object involved in the activity. Information visualisation can allow the student to access mathematical knowledge integrating the symbolic re-constructive approach with a motor-perceptive one. This latter approach involves actions and perceptions and produces learning based on doing, touching, moving and seeing. As already pointed out by Kaput [4[, a visual representation system based on direct manipulation interface is an interactive medium which responds to the user's action and which offers the possibility to create new notational systems or to introduce a new dimension, movement, within traditional ones.

Information visualisation offers the concrete possibility to implement better and more easily classical visual representation mathematics formats (graphs, drawings, tables, etc.) but also to enrich them with additional features such as movement and interactivity and to integrate in the same environment (or in interconnected environments) multiple formats of representations.

Within the context of use of the system, information visualisation can support different didactical functions which are crucial in the teaching and learning processes of mathematics.

Information visualisation can offer ways which allow students to explore the knowledge domain, embedded in the system (exploratory function). Systems that present these features have been demonstrated to be very effective for learning mathematics: they are defined as microworld based systems. The pedagogical objective of microworlds is to offer students a space in which they can use visualisation supported by the computer in order to explore and manage freely an environment designed to address the construction of some mathematical knowledge. Other systems that present these features are systems for simulation.

Another important didactical function of information visualisation is to offer expressive ways to allow students to externalise their own knowledge of a domain (expressive function). This function is present when representative tools, which student can easily control both on an operative and a conceptual level, are made available within a system.

A third didactical function regards the possibility to offer ways to validate the solution strategy involved in the problem at hand (validation function). In this case information visualisation makes available verification methods that are checked on a perceptive level and are able to test the strategy developed.

Systems that present this didactical function make available tools that allow students to justify their solution procedures on the basis of verification methods that are internal to the system.

A fourth didactical function is connected to the possibility to have available tools that allow the student to store the solution steps performed in order to successively re-visualise them in sequence. In this way it is possible to re-construct the story of the solution process performed. Systems which make available this kind of tools make possible to objectify, de-personalise and de-contextualize a solution procedure in order to successively transform it into an object of discussion in the social context of the class (supporting mathematical discourse function). This discussion can have different aims such as the comparison of strategies, the analysis of the properties (and of the their relationships) involved in the procedures, the re-interpretation of such properties in relation to a theory of reference. According to the nature of the knowledge domain involved in learning, the different described didactical functions of information visualisation can be pursued in different ways. In the following, we will try to better explicit how information visualisation can affect learning mathematics, analysing these different functions in a specific domain of knowledge supported by computer.

4. Information visualisation and learning Euclidean geometry

Geometry is a knowledge domain in which visualisation within a microworlds has brought to impressive progress with the development of the concept of “dynamic geometry” exemplified by systems such, for example, Cabri Gèométre. The researches worked out in relation to the use of this system put in evidence that its mediation can modify the way to enter into contact with geometrical knowledge. The changes brought into the didactical activity by the use of Cabri can be debated in terms of:

* New status assumed by the geometrical construction as mediated by the system;

* New possibilities of development of a mental image of a geometrical construction, theorem and proof, as mediated by the system.

In this framework we note that the first aspect can be put into relationship with the mapping (as reported in section 1) from the mathematical structure to the visual representation while the second aspect concerns the mapping from the visual representation to the mental image of geometrical construction, theorem and proof.

The relationship between geometrical construction and visual representation available with Cabri is particularly interesting.

By means of Cabri it is possible to perform every types of geometrical construction, within Euclidean geometry. Performing a geometrical construction by means of Cabri does not implicate aspects of measure, but it is strictly connected with the deep structure of Euclidean geometry. The fundamental element of every construction is the point. Some objects are directly defined in terms of points (i.e. a straight line is defined by means of two points). Other objects are defined as functions of some other objects already constructed (i.e. a straight line passing for a point perpendicular to a given line) [7].

Cabri makes available primitives deeply linked with the axioms of the Euclidean geometry but also a new visual direct manipulation opportunity: to drag the variable elements of the geometrical construction on the screen. In this way students can observe which properties are preserved when the construction is modified with the drag action. The movement produced by the drag action is a way to externalise the set of the relationships that define a figure.

These representation tools of Cabri can be used in both exploratory and expressive modes.

In the exploratory modes, the use of these representation tools containing domain models permit the student to examine the consequences of different model, some of which fit and other of which conflict with the student own ideas [9].

In the expressive mode student can examine their own knowledge of geometrical construction by being encouraged to experiment with their own theorizing [9].

Mariotti [8] highlights that the drag action of Cabri has a crucial importance both from a didactical and an epistemological point of view. By means of this action students can validate the constructions performed; the construction task is solved when the drawing get through the drag test, that is when the properties of the figure, under the drag action, are preserved (validation function).

Moreover the use of this new dimension, the movement, in the body of Euclidean geometry, makes available new visual tools to overcome epistemological obstacles connected with the relationship between drawing and figure in the domain of Euclidean geometry.

As stressed by Laborde [7], the drawing refers to a material entity with which one has a perceptive relation, figures refer to theoretical objects, that is to abstract geometrical object that are described by texts which define them.

In traditional didactical practice we note that several difficulties emerge in the geometrical construction tasks. Students work on the material drawing while the teacher expects they work on the figure or on the description of the figure. In other cases students do not consider that a geometrical construction task can involve the use of geometrical properties; rather they intend it as the request to produce a material visually correct drawing.

According to this framework, the opportunity of dragging the variable elements of a construction is an important tool for favouring the evolution of students' mental images, which are related, with the notion of geometrical figure and with the task itself of geometrical construction (exploratory and expressive function)

As an example the way in which student's approach geometrical constructions with Cabri can be considered. At the beginning students work on the drawing on the basis of perceptual and figural stimuli received from the screen instead of to perform a conceptual control of the perception, exploiting the action possibilities offered by the system. Only when they realise that their empirical strategy does not get through the dragging test, students, in general, begin to approach the task by using constructive methods which allow to maintain the properties of the figure when the drawing is dragged on the screen. It is in this moment that the relationship between figure and drawing is going to be modified and the student elaborates a new mental image of geometrical constructions.

The evolution of the mental image of geometrical constructions is crucial in geometry. As the matter of fact, each geometrical construction incorporates a theoretical meaning, which goes beyond the practical task of its construction. Mariotti in [8] shows that there is always a correspondence between the specific tools and rules used for the construction and a set of axioms which structure a part of a theory. Inside this theory a valid construction can always be put in correspondence with a theorem

Geometrical learning needs to move from context-based methods of validation (e.g. the dragging action of Cabri) to theory-based justifications (deductive method). This passage is neither simple nor automatic. It requires an educational approach able to support the acquisition of the deductive method. In this framework it is important specify the role that Cabri can assume in supporting this didactical approach.

In Cabri, for example, an important command is available at this regard: the "History command". This command permits to visualise step by step on the screen all the actions performed by the user to obtain a given geometrical construction. Such a command can develop some important function from a didactical point of view.

As put in evidence by Mariotti [8], the History command allows to develop interpretations and anticipation acts which can favour the passage from justifications based on physical actions and graphical effects to justifications based on a deductive model. In this passage there is the development of the capability to justify the validity of a procedure at the conceptual level (demonstration) on the basis of initial hypothesis, geometrical axioms and previously demonstrated theorems.

This passage cannot be spontaneously built as the result of the student-software interaction (e.g. as the result of having observed on the screen constructions that can be modified preserving some properties). The deductive method is built in the social interaction when students can experiment, under the teacher's guidance, the importance of justifying the accomplished procedure according to methods, which go beyond perceptual validation.

In the Cabri environment the History command is a fundamental tool at this regard. With this command the accomplished procedure becomes objective, de-personalised and de-contextualised.

In this way it is possible to focalise the attention on the properties of the construction and on their relationships and to consider them in the light of a geometrical theory and of previously demonstrated facts.

The History command is an important tool to maintain the mathematical discourse, which is developed in the class on the value, methods and techniques of geometrical demonstration integrated and "in focus" (supporting mathematical discourse function). This can be obtained thanks to the possibility offered of analysing and putting in relation the accomplished constructions

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